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OPTIMIZATION OF ELASTIC FOUNDATION FOR MINIMUM BEAM DEFLECTION

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Abstract—Beams attached to non-uniform elastic foundations and/or a set of concentrated springs are considered. The total stiffness of the foundation and springs is specified. The foundation stiffness distribution and the spring stiffnesses and locations are the design variables, and a measure of the beam deflection under given loading is minimized. Optimality conditions are derived using the calculus of variations. A number of examples are presented which involve a uniform cantilever, a uniformly distributed load, and a foundation with piecewise-constant stiffness. The optimal solutions often produce a significant decrease in the deflection measure in comparison with a reference design.

INTRODUCTION

This paper considers elastic beams supported in part by a non-uniform elastic foundation, or a set of concentrated springs, or both. The total stiffness of the foundation and springs is specified, but the foundation stiffness distribution and the spring stiffnesses and locations are variable. The objective is to minimize a measure of the beam deflection under given loads.

In Refs [1–17], the locations of rigid or flexible supports were optimized. Beam vibrations were treated in Refs [1–5]. If the fundamental natural frequency is to be maximized, the optimal location of a simple support is at the node of the second mode of the original beam, while the optimal location of a flexible support is the same or is nearby. In Ref. [5], a harmonically-varying load was applied to a cantilevered beam and the dynamic compliance was minimized. The support with variable location was either rigid or comprised of two springs and a mass in series.

Design for minimum compliance was examined in Refs [3, 6–9]. Both rigid and flexible supports were used. In Refs [6, 7, 10, 11], plastic structures were considered. In particular, Ref. [11] treated a circular plate with a given number of point supports; the support locations were chosen to maximize the load carrying capacity. Another plate problem was analyzed in Ref. [12], where the maximum elastic deflection was minimized. In Ref. [13], minimization of the maximum bending moment in a beam was investigated.

Stability was examined in Refs [3, 14–17]. As in the case of vibrations, a rigid support often should be placed at the node of the second mode. More than one support was included in Ref. [16], with a cost associated with each support. A flexible support was considered in Ref. [3], and its optimal location depended on the support stiffness. In Ref. [17], a cantilevered column was subjected to a follower load and the position of a dashpot was chosen to maximize the critical load.

An example involving optimization of the stiffnesses of discrete, flexible supports was analyzed in Refs [18–20]. A horizontal bar was supported by vertical springs at its ends (with stiffnesses c_1) and at its center (with stiffness c_2), forming a two-degrees-of-freedom system. A linear combination of the stiffnesses, $a_1c_1+a_2c_2$, was minimized subject to a

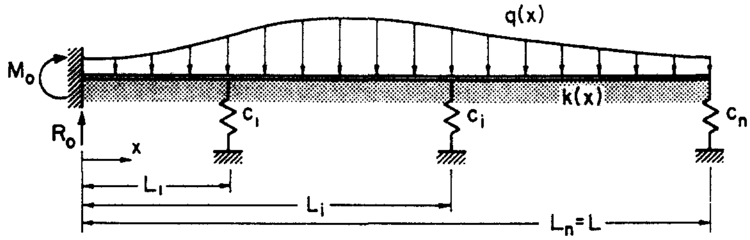

Fig. 1. Geometry of system (in dimensional terms).

minimum allowable value for the two natural frequencies. The solution was bimodal for a certain range of a_1/a_2.

Finally, Refs [4, 21] optimized the stiffness distribution of an elastic foundation. In Ref. [4], a uniform, pinned–pinned beam attached to a Winkler foundation with stiffness $k(x)$ per unit length was considered. The foundation inertia was neglected. For a specified fundamental frequency of free vibrations of the beam, the design function $k(x)$ was determined which minimized the total foundation stiffness. Two types of optimal solutions were obtained, depending on the value of the fundamental frequency. In one solution, all the stiffness was concentrated in a flexible support at the center of the span. The other solution consisted of two symmetrically-placed flexible supports and a uniform foundation between them.

In Ref. [21], a beam resting on a non-uniform foundation was displaced downwards at its ends. The stiffness distribution $k(x)$ was chosen to minimize the maximum foundation pressure $k(x)w(x)$. Constraints on total foundation stiffness and on maximum stiffness k_{max} were incorporated. In the optimal solution, $k(x)$ was equal to k_{max} in the central portion, while the pressure $k(x)w(x)$ was constant in the remaining outer regions of the beam.

This paper considers a general formulation, including a non-uniform elastic foundation of the Winkler type and a set of concentrated springs. The stiffness distribution of the foundation, the stiffnesses of the springs, and the locations of the springs are design variables, with the total stiffness specified. A measure of the deflection of an elastic beam under given loads is chosen as the objective functional to be minimized. Optimality criteria are derived with the use of the calculus of variations.

Numerical results are presented for several examples involving a uniform cantilevered beam subjected to a uniformly distributed load. First, two springs are attached to the beam, one at its tip and the other at midspan, and their stiffnesses are optimized. Then the location of the internal spring is also varied and the optimal solution is determined. Next, an elastic foundation with constant stiffness over each half of the beam is considered. Then the two segment lengths are allowed to vary, as well as the stiffness in each segment. Examples of piecewise-constant foundations with four and 100 equal-length segments are also treated, and finally the case of two foundation segments of variable lengths, separated by a concentrated spring, is analyzed.

FORMULATION

Consider an elastic beam of length L and continuous bending stiffness $EI(x)$ which is attached to an elastic foundation of stiffness $k(x)$ and elastic springs with stiffnesses c_i at locations $x = L_i$, $i = 1, \ldots, n$, as shown in Fig. 1. The foundation stiffness $k(x)$ is assumed to be continuous except for possible jumps at $x = L_i$. (This does not restrict discontinuities of k to locations of springs, since c_i can be set equal to zero.)

The beam is subjected to a continuous, distributed, transverse load $q(x)$. It is assumed that the end $x = 0$ is clamped, with reactions M_0 and R_0, and the end $x = L$ is attached to a spring (i.e. $L_n = L$). (A change in support conditions would only require minor modifications of the subsequent analysis.)

The bending moment, shear force, and deflection fields within the beam are denoted by $M(x)$, $V(x)$, and $w(x)$, respectively, with w positive if downward. The reactions M_0 and

R_0 in Fig. 1 indicate positive senses for M and V, respectively. Then the usual field equations are

$$M = -EIw'', \qquad V = -(EIw'')', \qquad (EIw'')''+kw = q \tag{1}$$

with boundary conditions

$$w = 0, \quad w' = 0 \quad \text{at} \quad x = 0; \qquad M = 0, \quad V = -c_n w \quad \text{at} \quad x = L \tag{2}$$

and transition conditions

$$w, w', M \text{ continuous}; \quad V^- - V^+ = -c_i w \quad \text{at} \quad x = L_i \quad (i = 1, \ldots, n-1) \tag{3}$$

where minus and plus superscripts denote values just to the left and right of a given point, respectively.

The objective functional is chosen as

$$G = \left\{ \int_0^L w^r(x)\, \mathrm{d}x \right\}^{1/r} \tag{4}$$

where r is a positive integer and should be even if the deflection is upward (i.e. $w < 0$). If $r = 1$, G is the compliance (i.e. the work done by the load). If r is large, G is an approximation to the maximum deflection of the beam[22]. The total stiffness of the foundation and springs is specified to have the value K_T, so that

$$\int_0^L k(x)\, \mathrm{d}x + \sum_{i=1}^{n} c_i = K_T. \tag{5}$$

In addition, the following constraints are included: for $i = 1, \ldots, n$

$$k(x) \geqslant k_{\min} \quad \text{for} \quad L_{i-1} < x < L_i; \quad c_i \geqslant c_{\min}; \quad L_i \geqslant L_{i-1} \tag{6}$$

where $k_{\min}$ and $c_{\min}$ are non-negative constants, $L_0 = 0$, and $L_n = L$. The design variables are $k(x)$, c_i $(i = 1, \ldots, n)$, and L_i $(i = 1, \ldots, n-1)$.

In order to derive the optimality conditions for minimization of G subject to eqns (1), (5), and (6), the following augmented functional G^* is constructed:

$$\begin{aligned} G^* = {} & \sum_{i=1}^{n} \int_{L_{i-1}}^{L_i} w^r\, \mathrm{d}x + \sum_{i=1}^{n} \int_{L_{i-1}}^{L_i} \lambda[-(EIw'')''-kw+q]\, \mathrm{d}x \\ & + \mu\left\{ \sum_{i=1}^{n} \int_{L_{i-1}}^{L_i} k\, \mathrm{d}x + \sum_{i=1}^{n} c_i - K_T \right\} + \sum_{i=1}^{n} \int_{L_{i-1}}^{L_i} \beta(k_{\min}-k+\theta^2)\, \mathrm{d}x \\ & + \sum_{i=1}^{n} \gamma_i(c_{\min}-c_i+\phi_i^2) + \sum_{i=1}^{n} \alpha_i(L_{i-1}-L_i+\psi_i^2) \end{aligned} \tag{7}$$

where $\lambda(x)$, μ, $\beta(x)$, γ_i, and α_i are Lagrange multipliers, and $\theta(x)$, ϕ_i, and ψ_i are slack variables. The stationarity condition $\delta G^* = 0$ then yields

$$\sum_{i=1}^{n}\int_{L_{i-1}}^{L_i}[-(EIw'')''-kw+q]\delta\lambda\,\mathrm{d}x+\sum_{i=1}^{n}\int_{L_{i-1}}^{L_i}[-(EI\lambda'')''-k\lambda+rw^{r-1}]\delta w\,\mathrm{d}x$$
$$+\sum_{i=1}^{n}[\lambda\,\delta V-\lambda'\,\delta M+M^a(\delta w)'-V^a\,\delta w]_{L_{i-1}}^{L_i}+\sum_{i=1}^{n}\int_{L_{i-1}}^{L_i}(-\lambda w+\mu-\beta)\,\delta k\,\mathrm{d}x$$
$$+\sum_{i=1}^{n}(\mu-\gamma_i)\,\delta c_i+\mu\sum_{i=1}^{n-1}[k^-(L_i)-k^+(L_i)]\,\delta L_i+\sum_{i=1}^{n}\alpha_i(\delta L_{i-1}-\delta L_i)$$
$$+\left[\sum_{i=1}^{n}\int_{L_{i-1}}^{L_i}k\,\mathrm{d}x+\sum_{i=1}^{n}c_i-K_T\right]\delta\mu+\sum_{i=1}^{n}\int_{L_{i-1}}^{L_i}(k_{\min}-k+\theta^2)\,\delta\beta\,\mathrm{d}x$$
$$+2\sum_{i=1}^{n}\int_{L_{i-1}}^{L_i}\beta\theta\,\delta\theta\,\mathrm{d}x+\sum_{i=1}^{n}(c_{\min}-c_i+\phi_i^2)\,\delta\gamma_i+2\sum_{i=1}^{n}\gamma_i\phi_i\delta\phi_i$$
$$+\sum_{i=1}^{n}(L_{i-1}-L_i+\psi_i^2)\,\delta\alpha_i+2\sum_{i=1}^{n}\alpha_i\psi_i\delta\psi_i=0 \qquad (8)$$

where $M^a=-EI\lambda''$ and $V^a=-(EI\lambda'')'$.

Due to eqns (1), the first term in eqn (8) is zero. The adjoint variable $\lambda(x)$ is chosen to satisfy the equation

$$(EI\lambda'')''+k\lambda=rw^{r-1} \qquad (9)$$

and the same conditions (2) and (3) as $w(x)$, so that the second term in eqn (8) also vanishes. With the use of eqns (1)–(3), together with eqns (A7) and (A12) from the Appendix, the third term in eqn (8) becomes

$$\sum_{i=1}^{n-1}\{-c_i\lambda'(L_i)w(L_i)-c_i\lambda(L_i)w'(L_i)-[k^-(L_i)-k^+(L_i)]\lambda(L_i)w(L_i)\}\,\delta L_i-\sum_{i=1}^{n}\lambda(L_i)w(L_i)\,\delta c_i. \qquad (10)$$

It then follows from eqn (8) that the optimal solution is governed by eqns (1)–(3), (5), (9), and the conditions

$$-c_i\lambda'(L_i)w(L_i)-c_i\lambda(L_i)w'(L_i)-[k^-(L_i)-k^+(L_i)][\lambda(L_i)w(L_i)-\mu]+\alpha_{i+1}-\alpha_i=0 \quad (i=1,\ldots,n-1) \qquad (11)$$

$$L_{i-1}-L_i+\psi_i^2=0,\qquad \alpha_i\psi_i=0\quad (i=1,\ldots,n) \qquad (12)$$

$$-\lambda(L_i)w(L_i)+\mu-\gamma_i=0\quad (i=1,\ldots,n) \qquad (13)$$

$$c_{\min}-c_i+\phi_i^2=0,\qquad \gamma_i\phi_i=0\quad (i=1,\ldots,n) \qquad (14)$$

$$-\lambda(x)w(x)-\beta(x)+\mu=0\quad (L_{i-1}<x<L_i;\ i=1,\ldots,n) \qquad (15)$$

$$k_{\min}-k(x)+\theta^2(x)=0,\quad \beta(x)\theta(x)=0\quad (L_{i-1}<x<L_i;\quad i=1,\ldots,n). \qquad (16)$$

If the locations L_i of springs or foundation discontinuities are fixed and not included among the design variables, eqns (11) and (12) are not applicable. If the concentrated springs are not included in the problem, eqns (13) and (14) are neglected. If there is no elastic foundation, eqns (15) and (16) are dropped. If $k(x)$ is assumed to be piecewise constant with $k(x)=k_i$ for $L_{i-1}<x<L_i$ $(i=1,\ldots,n)$, then eqns (15) and (16) are replaced by

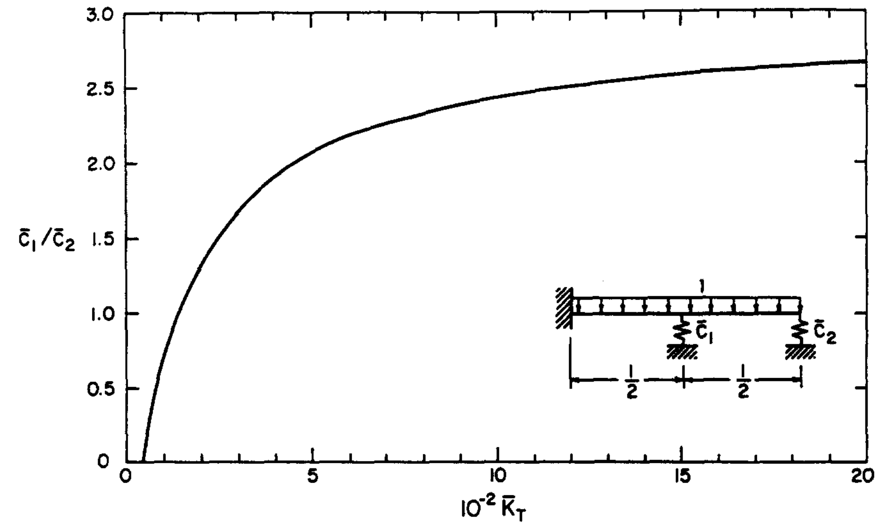

Fig. 2. Optimal spring stiffness ratio for Example 1.

$$-\int_{L_{i-1}}^{L_i} \lambda(x)w(x)\,dx+(L_i-L_{i-1})(\mu-\beta_i)=0,\quad k_{\min}-k_i+\theta_i^2=0,\quad \beta_i\theta_i=0\quad (i=1,\ldots,n). \tag{17}$$

EXAMPLES

In the examples, the beams are fixed at $x = 0$ and free at $x = L$, the bending stiffness EI and the distributed load q are constant, the foundation stiffness is piecewise constant, and $k_{\min}$ and $c_{\min}$ are set equal to zero. The following dimensionless quantities are introduced:

$$\xi = x/L,\quad l_i = L_i/L,\quad \bar{w} = (EI/qL^4)w,\quad \bar{\lambda} = (EI/qL^4)\lambda,$$
$$\bar{k}_i = (L^4/EI)k_i,\quad \bar{c}_i = (L^3/EI)c_i,\quad \bar{K}_T = (L^3/EI)K_T \tag{18}$$

along with appropriate dimensionless Lagrange multipliers $\bar{\mu}$, $\bar{\beta}$, $\bar{\beta}_i$, $\bar{\gamma}_i$, and $\bar{\alpha}_i$, and slack variables $\bar{\theta}$, $\bar{\theta}_i$, $\bar{\phi}_i$, and $\bar{\psi}_i$.

Example 1

For the first example, consider the beam shown in Fig. 2 (in dimensionless terms). The design variables are $\bar{c}_1$ and $\bar{c}_2$. If the objective functional is chosen as eqn (4) with $r = 1$ (i.e. the compliance is to be minimized), the governing equations, eqns (1) and (9), for the deflection $\bar{w}(\xi)$ and adjoint variable $\bar{\lambda}(\xi)$ become

$$\bar{w}''''(\xi) = 0,\qquad \bar{\lambda}''''(\xi) = 0. \tag{19}$$

The optimality conditions (13) and (14) take the form

$$-\bar{\lambda}(1/2)\bar{w}(1/2)+\bar{\mu}-\bar{\gamma}_1=0,\quad -\bar{\lambda}(1)\bar{w}(1)+\bar{\mu}-\bar{\gamma}_2=0,\quad -\bar{c}_i+\bar{\phi}_i^2=0,\quad \bar{\gamma}_i\bar{\phi}_i=0\quad (i=1,2) \tag{20}$$

while the constraint (5) of specified total stiffness becomes

$$\bar{c}_1+\bar{c}_2=\bar{K}_T. \tag{21}$$

If $\bar{c}_1 > 0$ and $\bar{c}_2 > 0$, the solution is given by

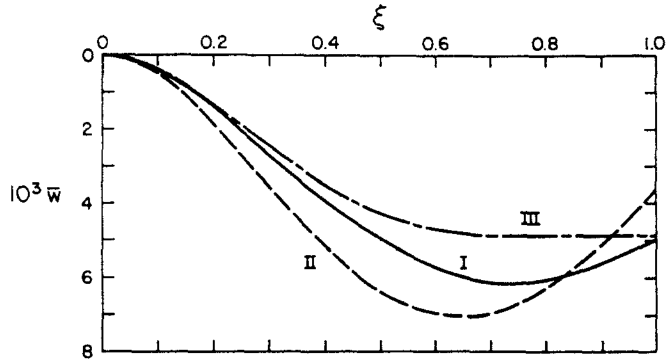

Fig. 3. Beam deflections for $\bar{K}_T = 100$: I, optimal solution of Example 1; II, reference case $\bar{c}_1 = 0$, $\bar{c}_2 = \bar{K}_T$; III, optimal solution of Example 2.

$$24\bar{w}(\xi) = 24\bar{\lambda}(\xi) = \begin{cases} 6(1-b_1-2b_2)\xi^2-4(1-b_1-b_2)\xi^3+\xi^4 & \text{for} \quad 0 \leqslant \xi \leqslant \frac{1}{2} \\ 6(1-b_1-2b_2)\xi^2-4(1-b_1-b_2)\xi^3+\xi^4-4b_1(\xi-\frac{1}{2})^3 & \\ \text{for} \quad \frac{1}{2} \leqslant \xi \leqslant 1 & \end{cases} \tag{22}$$

where

$$b_1 = 2\bar{c}_1(51+2\bar{c}_2)/h, \qquad b_2 = \bar{c}_2(2304+11\bar{c}_1)/(8h), \qquad h = 2304+96\bar{c}_1+768\bar{c}_2+7\bar{c}_1\bar{c}_2 \tag{23}$$

with optimal spring stiffnesses

$$\bar{c}_1 = (32\bar{K}_T-1488)/43, \qquad \bar{c}_2 = (1488+11\bar{K}_T)/43. \tag{24}$$

It is noted that $\bar{w}(1/2) = \bar{w}(1)$ and $b_1/\bar{c}_1 = b_2/\bar{c}_2$. This solution is valid for $\bar{K}_T \geqslant 46.5$. If $0 \leqslant \bar{K}_T < 46.5$, the optimal stiffnesses are $\bar{c}_1 = 0$, $\bar{c}_2 = \bar{K}_T$.

In Fig. 2, the ratio $\bar{c}_1/\bar{c}_2$ of the optimal stiffnesses is plotted as a function of the dimensionless total foundation stiffness $\bar{K}_T$. As $\bar{K}_T \to \infty$, this ratio approaches the value 2.909. For $\bar{K}_T = 100$, Fig. 3 shows the beam deflection for the optimal ratio $\bar{c}_1/\bar{c}_2 = 0.66$ as curve I and for the reference case $\bar{c}_1 = 0$, $\bar{c}_2 = \bar{K}_T$ as curve II. The compliance is 11.9% lower for curve I than curve II, and it turns out that the maximum deflection is 12.2% lower for curve I than curve II at this value of $\bar{K}_T$.

Example 2

Consider the same beam as in Example 1, but here the location $\xi = l_1$ of the internal spring is added to the design variables $\bar{c}_1$ and $\bar{c}_2$ (see Fig. 4). Again, the compliance is to be minimized (i.e. $r = 1$ in eqn (4)).

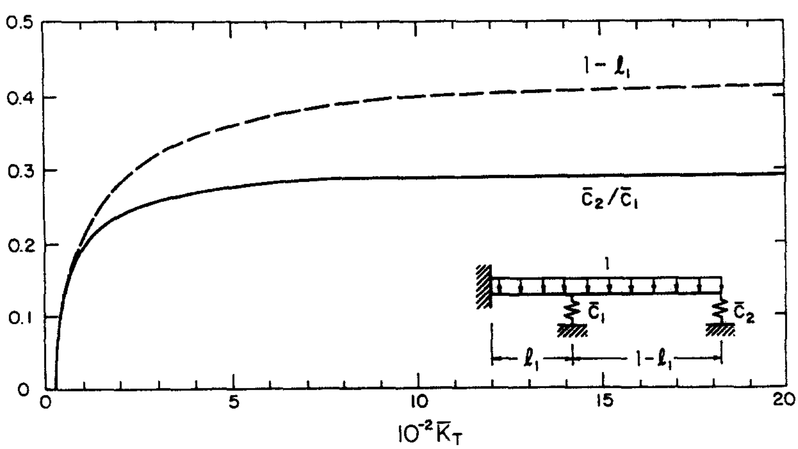

Fig. 4. Optimal spring stiffness ratio and segment length for Example 2.

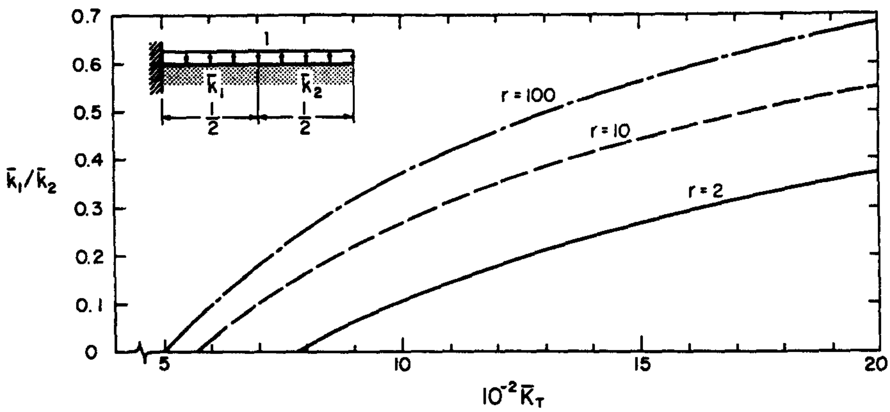

Fig. 5. Optimal foundation stiffness ratios for Example 3.

The optimality conditions are given by eqns (20), with 1/2 replaced by l_1, and also, from eqns (11) and (12)

$$-\bar{c}_1\bar{\lambda}'(l_1)\bar{w}(l_1)-\bar{c}_1\bar{\lambda}(l_1)\bar{w}'(l_1)+\bar{\alpha}_2-\bar{\alpha}_1=0, \quad -l_1+\bar{\psi}_1^2=0, \quad l_1-1+\bar{\psi}_2^2=0,$$
$$\bar{\alpha}_1\bar{\psi}_1=0, \quad \bar{\alpha}_2\bar{\psi}_2=0. \tag{25}$$

Constraint (21) is still applicable. If $\bar{c}_1>0$, $\bar{c}_2>0$, and $0<l_1<1$, the optimal deflections have the form (22) if 1/2 is replaced by l_1, and if b_1 and b_2 are given by

$$b_1=[2\bar{c}_1(3+\bar{c}_2)(6-4l_1+l_1^2)l_1^2-3\bar{c}_1\bar{c}_2(3-l_1)l_1^2]/h,$$
$$b_2=[6\bar{c}_2(3+\bar{c}_1l_1^3)-\bar{c}_1\bar{c}_2(3-l_1)(6-4l_1+l_1^2)l_1^4]/h,$$
$$h=16(3+\bar{c}_2)(3+\bar{c}_1l_1^3)-4\bar{c}_1\bar{c}_2(3-l_1)^2l_1^4 \tag{26}$$

and the optimal values of $\bar{c}_1$, $\bar{c}_2$, and l_1 are obtained with the use of eqn (21) and the equations

$$b_1/\bar{c}_1=b_2/\bar{c}_2, \qquad \bar{w}'(l_1)=0. \tag{27}$$

The first condition in eqns (27) follows from the fact that $\bar{w}(l_1)=\bar{w}(1)$. If $\bar{K}_T<24$, the optimal solution has $l_1=1$, i.e. there is a single spring at the tip of the cantilever.

The optimal ratio $\bar{c}_2/\bar{c}_1$ and length $1-l_1$ are plotted in Fig. 4 as functions of $\bar{K}_T$. As $\bar{K}_T\to\infty$, one finds that $\bar{c}_2/\bar{c}_1\to 0.30$ and $l_1\to 0.55$. For the value $\bar{K}_T=100$, the optimal solution gives $\bar{c}_2/\bar{c}_1=0.19$, $l_1=0.79$, and the beam deflection III drawn in Fig. 3. The compliance is 22.2% lower for curve III than curve II (the reference case $\bar{c}_1=0$, $\bar{c}_2=\bar{K}_T$) and the maximum deflection is 30.1% lower for curve III than curve II at this value of $\bar{K}_T$.

Example 3

In this example, the beam is supported by a foundation with dimensionless stiffness $\bar{k}_1$ for $0<\xi<1/2$ and $\bar{k}_2$ for $1/2<\xi<1$ (see Fig. 5). From eqns (1) and (9), it follows that

$$\bar{w}''''(\xi)+\bar{k}_i\bar{w}(\xi)=1, \quad \bar{\lambda}''''(\xi)+\bar{k}_i\bar{\lambda}(\xi)=r\bar{w}^{r-1}(l_{i-1}<\xi<l_i;\ \ i=1,2) \tag{28}$$

where $l_0=0$, $l_1=1/2$, and $l_2=1$. From eqn (5)

$$l_1\bar{k}_1+(1-l_1)\bar{k}_2=\bar{K}_T. \tag{29}$$

If $\bar{k}_1=0$, then $\bar{k}_2=2\bar{K}_T$. If $\bar{k}_1>0$ and $\bar{k}_2>0$, then eqns (17) lead to the optimality condition

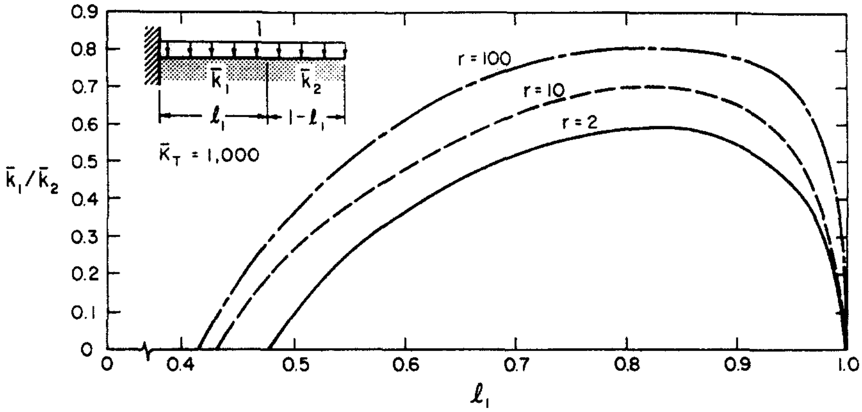

Fig. 6. Optimal foundation stiffness ratios for $\bar{K}_T = 1000$ in Example 4.

$$\int_0^{l_1} \bar{\lambda}(\xi)\bar{w}(\xi)\, \mathrm{d}\xi = \int_{l_1}^{1} \bar{\lambda}(\xi)\bar{w}(\xi)\, \mathrm{d}\xi. \tag{30}$$

The finite difference method was applied to eqns (28), Simpson's rule was applied to eqn (30), and a non-linear equation solver was used to find numerical solutions of the resulting set of equations.

Values $r = 2$, 10, and 100 were chosen in the objective functional (4). The resulting optimal ratios $\bar{k}_1/\bar{k}_2$ are plotted in Fig. 5. The foundation stiffness $\bar{k}_1$ vanishes if $\bar{K}_T$ is sufficiently small, i.e. if $\bar{K}_T < \bar{K}_0$ where $\bar{K}_0 = 782.7$ for $r = 2$, $\bar{K}_0 = 575.4$ for $r = 10$, and $\bar{K}_0 = 502.9$ for $r = 100$. As $\bar{K}_T \to \infty$, $\bar{k}_1/\bar{k}_2 \to 1$. In comparison to a uniform foundation, the optimal value of the objective functional G at $\bar{K}_T = \bar{K}_0$ is lower by 23.3% if $r = 2$, 30.7% if $r = 10$, and 34.6% if $r = 100$. These percentages decrease as $\bar{K}_T$ increases.

Example 4

This example is similar to the previous one, with two segments of constant foundation stiffness, but the lengths of the segments are not equal. The value of l_1 is fixed, and the governing equations are still eqns (28)–(30). Results are plotted in Fig. 6 for total foundation stiffness $\bar{K}_T = 1000$ and for values $r = 2$, 10, and 100 in eqn (4). If l_1 is sufficiently small, the optimal value of $\bar{k}_1$ is zero, and the optimal ratio $\bar{k}_1/\bar{k}_2$ possesses a maximum when l_1 is approximately 0.83.

Example 5

As in Examples 3 and 4, the foundation here is piecewise constant in two segments. However, the lengths of the segments are not specified, but are allowed to vary. It turns out that the set of eqns (11), (12), (17), (28), and (29) does not have a solution with $\bar{k}_1$ and $\bar{k}_2$ nonzero, and that $\bar{k}_1 = 0$ in the optimal solution (see Fig. 7). If $\bar{K}_T$ is sufficiently

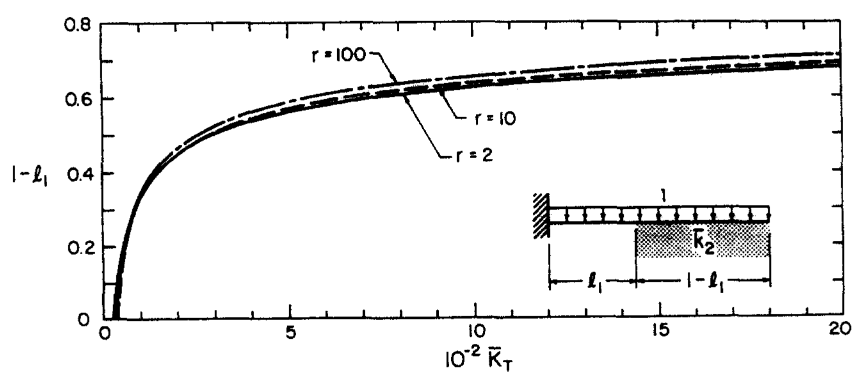

Fig. 7. Optimal segment lengths for Example 5.

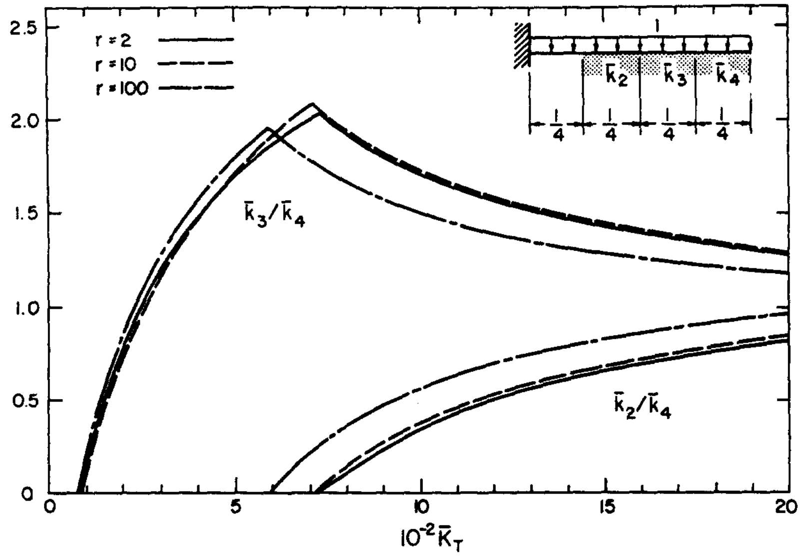

Fig. 8. Optimal foundation stiffness ratios for Example 6.

small, $l_1 = 1$ and the foundation reduces to a concentrated spring at the tip of the cantilever. Otherwise, the governing equations become eqns (28) and (29) with $\bar{k}_1 = 0$, and

$$\int_{l_1}^{1} \bar{\lambda}(\xi)\bar{w}(\xi)\,\mathrm{d}\xi = (1-l_1)\bar{\lambda}(l_1)\bar{w}(l_1). \tag{31}$$

The optimal lengths $1-l_1$ of the foundation segment with stiffness $\bar{k}_2 = \bar{K}_T/(1-l_1)$ are plotted in Fig. 7 for $r = 2$, 10, and 100 in eqn (4). At the value of $\bar{K}_T$ where $1-l_1$ increases from zero, the decrease of the objective functional from that for a uniform foundation over the whole beam is 55% if $r = 2$, 61% if $r = 10$, and 66% if $r = 100$. These percentages decrease as $\bar{K}_T$ increases.

Example 6

In this example, the foundation consists of four equal-length segments with constant stiffness in each one. From the numerical solution, one finds that $\bar{k}_1 = 0$. Optimal ratios $\bar{k}_2/\bar{k}_4$ and $\bar{k}_3/\bar{k}_4$ are shown in Fig. 8 for $r = 2$, 10, and 100. If $\bar{K}_T$ is sufficiently small, $\bar{k}_2$ and $\bar{k}_3$ are zero. As $\bar{K}_T$ increases, $\bar{k}_3$ becomes nonzero (at $\bar{K}_T = 79.9$, 83.0, and 76.5 for $r = 2$, 10, and 100, respectively), followed by $\bar{k}_2$ (at $\bar{K}_T = 725.4$, 707.1, and 591.5 for $r = 2$, 10, and 100, respectively). At the value of $\bar{K}_T$ where $\bar{k}_3$ becomes nonzero, the objective functional is lower than that for a uniform foundation by 48% if $r = 2$, 56% if $r = 10$, and 58% if $r = 100$. These percentages decrease as $\bar{K}_T$ increases.

Example 7

For this example, the foundation stiffness is constant in each of 100 equal-length segments. Equations (28) are satisfied in each segment and, if $\bar{k}_i > 0$, eqns (17) lead to

$$\int_{l_{i-1}}^{l_i} \bar{\lambda}(\xi)\bar{w}(\xi)\,\mathrm{d}\xi = \bar{\mu}/100. \tag{32}$$

Numerical results were obtained for $r = 2$ and values of $\bar{K}_T$ in the range 2200–3000. All the optimal solutions had a similar form. A typical case is $\bar{K}_T = 2300$, in which the optimal solution (to three significant digits) has $\bar{k}_i = 0$ for $1 \leqslant i \leqslant 39$, $\bar{k}_{40} = 16\,500$, $\bar{k}_{41} = 32\,600$, $\bar{k}_{42} = \bar{k}_{43} = 0$, $\bar{k}_{44} = 4490$, $\bar{k}_{45} = 3080$, $\bar{k}_{46} = 3210$, $\bar{k}_{47} = 3170$, $\bar{k}_{48} = 3180$, $\bar{k}_{49} = \bar{k}_{50} = 3170$, $\bar{k}_i = 3160$ for $51 \leqslant i \leqslant 55$, and $\bar{k}_i = 3150$ for $56 \leqslant i \leqslant 100$. This optimal

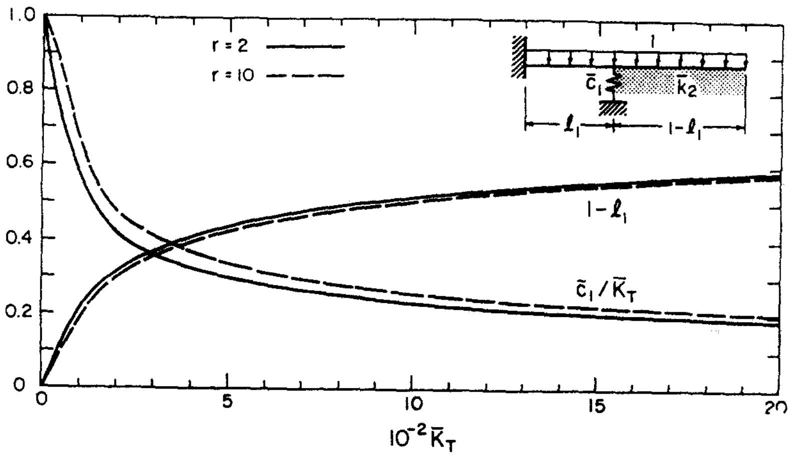

Fig. 9. Optimal spring stiffnesses and segment lengths for Example 8.

foundation can be approximated by a concentrated spring near $\xi = 0.40$ and a uniform foundation between the spring and the tip of the beam.

Example 8

This case is motivated by the results of Example 7 and by some optimal solutions presented in Refs [4, 7]. It is assumed that there is a spring of stiffness $\bar{c}_1$ at $\xi = l_1$ and a uniform foundation of stiffness $\bar{k}_2$ for $l_1 < \xi < 1$ (see Fig. 9). The design variables are $\bar{c}_1$, $\bar{k}_2$, and l_1, and they are related by the constraint

$$\bar{c}_1 + (1-l_1)\bar{k}_2 = \bar{K}_T \tag{33}$$

from eqn (5). Based on eqns (11)–(14) and (17), the optimality conditions reduce to eqn (31) and

$$\bar{\lambda}'(l_1)\bar{w}(l_1) + \bar{\lambda}(l_1)\bar{w}'(l_1) = 0. \tag{34}$$

Results are plotted in Fig. 9 for $r = 2$ and 10 in the objective functional G. As $\bar{K}_T \to 0$, one sees that $l_1 \to 1$. As $\bar{K}_T$ increases, the length $1-l_1$ of the foundation segment increases and the ratio $(1-l_1)\bar{k}_2/\bar{c}_1$ increases. In comparison to a uniform foundation, the optimal value of G is lower by 51% for $r = 2$ and 59% for $r = 10$ if $\bar{K}_T = 50$, and by 24% for $r = 2$ and 28% for $r = 10$ if $\bar{K}_T = 2000$.

CONCLUDING REMARKS

A given beam subjected to a given load was considered, and a measure of the beam deflection was minimized by determining optimal supporting springs and/or an elastic foundation with non-uniform stiffness distribution. The total stiffness of the springs and the foundation was specified. In the general formulation, a set of springs with arbitrary stiffnesses and locations was included, together with a foundation of arbitrary or piecewise-constant stiffness distribution. Optimality conditions were derived using the calculus of variations, and the solution procedure involved the finite difference method, numerical integration, and a non-linear equation solver.

Clamped-free boundary conditions were assumed, but the formulation can be modified easily for other cases. Numerical results were obtained for a uniform beam under a uniformly distributed load, with various special cases involving one or two concentrated springs and/or a piecewise-uniform foundation. In comparison to a reference case of a single spring at the tip or a uniform foundation, the optimal solution often leads to a significant decrease in the deflection measure.

Some further work has been carried out for the case $r = 1$ in eqn (4) and a uniformly distributed load[23]. Comparing eqns (1) and (9) for $r = 1$, one sees that $\lambda(x)$ is proportional to $w(x)$. Then, from eqns (15) and (16) with $k_{min} = 0$, it follows that $w^2(x)$ is constant when $k(x) > 0$, and, from eqns (1), $k(x)$ is constant. Concentrated springs separate regions of uniform foundation from regions of no foundation. Results are presented in Ref. [23] for beams with various boundary conditions and for a related problem involving circular plates.

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APPENDIX

Relations between variations of state fields and variations of the design parameters L_i are derived in this Appendix.

Consider a design k, c_i, L_i with the associated state field w, M, V, and a second design $k+\delta k$, $c_i+\delta c_i$, $L_i+\delta L_i$ with the associated state field $\hat{w}$, $\hat{M}$, $\hat{V}$ (see Fig. A1). Here δk, δc_i, and δL_i are infinitesimal quantities. It is assumed that the distributed load $q(x)$ and the bending stiffness $EI(x)$ are continuous. Then the deflections and their first two derivatives are continuous functions. All the following equations refer to location $x = L_i$ $(i = 1, \ldots, n-1)$.

From the continuity conditions, one has

$$w^- = w^+, \quad (w')^- = (w')^+, \quad (w'')^- = (w'')^+ \tag{A1}$$

and similarly for $\hat{w}$. One can write

$$\hat{w}(L_i+\delta L_i) = \hat{w}(L_i)+\hat{w}'\,\delta L_i = w(L_i)+\delta w+w'\,\delta L_i \tag{A2}$$

and

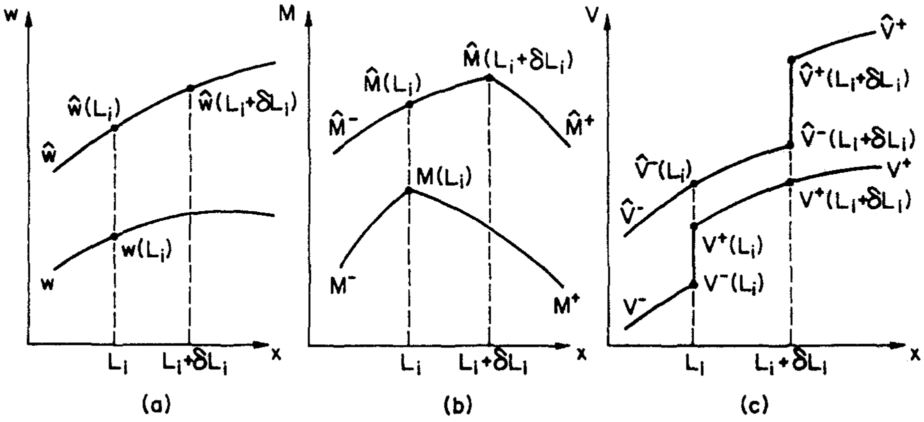

Fig. A1. Deflection, bending moment, and shear force for two designs.

$$\hat{w}'(L_i+\delta L_i) = \hat{w}'(L_i)+\hat{w}''\,\delta L_i = w'(L_i)+\delta w'+w''\,\delta L_i \tag{A3}$$

where $\delta w' \equiv (\delta w)'$. If eqn (A2) is applied on both sides of the spring and the resulting equations are subtracted from each other, and if the same is done for eqn (A3), one obtains

$$(\delta w)^- - (\delta w)^+ = 0, \quad (\delta w')^- - (\delta w')^+ = 0. \tag{A4}$$

For the bending moment, consider the relations

$$\hat{M}^-(L_i+\delta L_i) = \hat{M}^-(L_i)+(\hat{M}')^-\,\delta L_i = M^-(L_i)+\delta M^- + (M')^-\,\delta L_i \tag{A5}$$

$$\hat{M}^+(L_i+\delta L_i) = \hat{M}^+(L_i)+(\hat{M}')^+\,\delta L_i = M^+(L_i)+\delta M^+ + (M')^+\,\delta L_i. \tag{A6}$$

Subtracting eqn (A6) from eqn (A5) yields

$$\delta M^- - \delta M^+ = -[(M')^- - (M')^+]\,\delta L_i = -(V^- - V^+)\,\delta L_i = c_i w(L_i)\,\delta L_i \tag{A7}$$

with the use of eqns (1), (3), and (A1).

For the shear force

$$\hat{V}^-(L_i+\delta L_i) = \hat{V}^-(L_i)+(\hat{V}')^-\,\delta L_i = V^-(L_i)+\delta V^- + (V')^-\,\delta L_i \tag{A8}$$

$$\hat{V}^+(L_i+\delta L_i) = \hat{V}^+(L_i)+(\hat{V}')^+\,\delta L_i = V^+(L_i)+\delta V^+ + (V')^+\,\delta L_i. \tag{A9}$$

Also, from eqns (1), one can write

$$(V')^- - (V')^+ = (k^- w^- - q^-)-(k^+ w^+ - q^+) = (k^- - k^+)w(L_i). \tag{A10}$$

In addition, from eqns (3) and (A2) it follows that

$$\begin{aligned}\hat{V}^-(L_i+\delta L_i)-\hat{V}^+(L_i+\delta L_i) &= -(c_i+\delta c_i)(w+\delta w+w'\,\delta L_i)\\ &= -c_i w(L_i)-c_i\delta w-c_i w'(L_i)\,\delta L_i - w(L_i)\,\delta c_i.\end{aligned} \tag{A11}$$

Subtracting eqn (A9) from eqn (A8) and utilizing eqns (3), (A10), and (A11), one can obtain

$$\delta V^- - \delta V^+ = -c_i\delta w - w(L_i)\,\delta c_i - c_i w'(L_i)\,\delta L_i - [k^-(L_i)-k^+(L_i)]w(L_i)\,\delta L_i. \tag{A12}$$